

NO FINITE INVARIANT DENSITY FOR MISIUREWICZ EXPONENTIAL MAPS

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ABSTRACT. For exponential mappings such that the orbit of the only singular value 0 is bounded, it is shown that no integrable density invariant under the dynamics exists on \mathbb{C} .

1. INTRODUCTION

We consider one parameter family of exponential functions $f_\Lambda(z) = \Lambda e^z$, $z \in \mathbb{C}$, $\Lambda \in \mathbb{C}^*$. These maps have only one finite singular value 0 whose forward trajectory determines the dynamics on \mathbb{C} . From now we assume that the orbit of the asymptotic value 0 is bounded, hence the Julia set $J(f_\Lambda) = \mathbb{C}$. Thus f satisfies so called Misiurewicz condition i.e. the post-singular set $P(f) := \overline{\bigcup_{n=0}^{\infty} f_\Lambda^n(0)}$ is bounded and $P(f) \cap \text{Crit}(f) = \emptyset$. It follows from [5, Th.1] that $P(f)$ is hyperbolic. The problem of existence of probabilistic invariant measure absolutely continuous with respect to the Lebesgue measure (abbr. *pacim*) for transcendental meromorphic functions satisfying Misiurewicz condition was discussed in [7]. However this result cannot be applied to entire functions. It is still an open problem whether the simplest entire functions like exponential map $z \rightarrow 2\pi i \exp(z)$ have *pacim*. The main result of this paper is the following theorem.

Theorem 1. *Let $f(z) = \Lambda \exp(z)$ with $\Lambda \in \mathbb{C} \setminus \{0\}$ chosen so that the Julia set is the entire sphere and the orbit of 0 under f is bounded. Then f admits no probabilistic invariant measure absolutely continuous with respect to the Lebesgue measure.*

However these maps have σ -finite invariant measure absolutely continuous with respect to the Lebesgue measure (see [6]). A result similar to Theorem 1 has been mentioned to us by other authors, [3].

The proof will proceed by contradiction, so we suppose that such a measure exists and call it μ , while reserving λ for the Lebesgue measure of the plane. It follows from [4] that the set of points escaping to ∞ has zero Lebesgue's measure for every map in our family. It is not difficult to prove that for these functions the union $P(f) \cup \{\infty\}$ is not a metric attractor in sense of Milnor with respect to the measure λ on \mathbb{C} . The results of [1] implies that f_Λ is ergodic with respect to λ . Thus

Fact 1. *The measure μ is ergodic.*

2. PROOF

For a positive integer n write $A_n := \{z : |\Lambda|e^n < |z| \leq |\Lambda|e^{n+1}, \arg z \neq \arg \Lambda\}$. A *fundamental rectangle* will refer to any set in the form $\{x + 2\pi iy : k < x < k+1, l < y < l+1\}$ for integers k, l . Thus, any fundamental rectangle is mapped with bounded distortion and onto some A_n .

Lemma 1. *For all $n \in \mathbb{Z}_+$, $\inf \text{ess} \left\{ \frac{d\mu}{d\lambda}(z) : z \in A_n \right\} > 0$.*

Proof. By [5, Th.1], the post-singular set $P(f)$ has area 0, so it cannot be the support of μ . Additionally, the image of every open set covers A_n after finitely many iterations, so it suffices to have the $\frac{d\mu}{d\lambda}$ essentially bounded away from 0 on any open set. Hence, Lemma 1 follows from the following fact. \square

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Lemma 2. *Suppose that F is a meromorphic function whose Julia set is the entire sphere, and ν a probability invariant and ergodic measure absolutely continuous with respect to λ and such that the ν -measure of the closure of the post-singular set of F is less than 1. Then, there is an open set U such that*

$$\inf \operatorname{ess} \left\{ \frac{d\nu}{d\lambda}(z) : z \in U \right\} > 0.$$

Proof. Fix U to be a disk in a positive distance from the orbit of 0 and such that $\eta := \nu(U)$ is positive. Denote $\rho(z) := \frac{d\nu}{d\lambda}$. Pick $\epsilon > 0$. In the argument to follow it is important distinguish between parameters that do or do not depend on ϵ .

A variant of Luzin's Theorem. For every $\epsilon > 0$, we can find a continuous function with compact support $\rho_\epsilon : \mathbb{C} \rightarrow [0, +\infty)$ such that

$$(1) \quad \int_{\mathbb{C}} (\rho_\epsilon(w) - \rho(w))_+ d\lambda(w) < \epsilon$$

where the plus subscript denote the positive part,

$$(2) \quad \int_{\mathbb{C}} \min(\rho_\epsilon(z), \rho(z)) d\lambda(z) \geq 1 - \eta/10.$$

This statement follows from introductory measure theory.

Proof of Lemma 1 continued. Now for any k consider the set Ω_k of connected components of $F^{-k}(U)$ which intersect the support of ρ_ϵ . If $V \in \Omega_k$, then F^k maps V onto U univalently and with distortion bounded depending solely on U . Denote $d_k = \sup\{\operatorname{diam} V : V \in \Omega_k\}$. Since the Julia set is the whole sphere, $\lim_{k \rightarrow \infty} d_k = 0$. Let G_k denote the set of inverse branches of F^k defined on U . For z in U

$$\rho_{\epsilon,k}(z) = \sum_{g \in G_k} \inf\{\rho_\epsilon(w) : w = g(z), z \in U\} |g'(z)|^2.$$

For any g , the ratio of the values of each summand at two points z_1, z_2 is equal to the ratio of $|g'|^2$ at these points, hence bounded above by some $Q_0 \geq 1$ which depends solely on the distortion of inverse branches and therefore only on U . Consequently,

$$(3) \quad \frac{\rho_{\epsilon,k}(z_1)}{\rho_{\epsilon,k}(z_2)} \leq Q_0$$

for every $z_1, z_2 \in U$. Consider a similarly constructed

$$\tilde{\rho}_\epsilon(z) = \sum_{g \in G_k} \rho_\epsilon(g(z)) |g'(z)|^2.$$

By the change of variable formula

$$\begin{aligned} \int_U \tilde{\rho}_\epsilon(z) d\lambda(z) &= \int_{F^{-k}(U)} \rho_\epsilon(w) d\lambda(w) \geq \int_{F^{-k}(U)} \min(\rho(w), \rho_\epsilon(w)) d\lambda(w) = \\ &= \int_{\mathbb{C}} \min(\rho(w), \rho_\epsilon(w)) d\lambda(w) - \int_{F^{-k}(U)^c} \min(\rho(w), \rho_\epsilon(w)) d\lambda(w) \geq \\ (4) \quad &\geq 1 - \eta/10 - \nu(F^{-k}(U)^c) = 1 - \eta/10 - (1 - \eta) = \frac{9}{10}\eta \end{aligned}$$

where we have also used condition (2). Clearly, $\rho_{\epsilon,k} \leq \tilde{\rho}_\epsilon$. Let δ_ϵ denote the modulus of continuity of ρ_ϵ . Then

$$\int_U (\tilde{\rho}_\epsilon(z) - \rho_{\epsilon,k}(z)) d\lambda(z) \leq \delta_\epsilon(d_k) \int_U \sum_{g \in G'_k} |g'(z)|^2 d\lambda(z).$$

Here G'_k denoted the set of only those inverse branches which map onto some $V \in \Omega_k$. By bounded distortion, if g maps on V , then for any $z \in U$, $|g'(z)|^2 \leq Q_0 \frac{\lambda(V)}{\lambda(U)}$. Hence, we can further estimate

$$\int_U (\tilde{\rho}_\epsilon(z) - \rho_{\epsilon,k}(z)) d\lambda(z) \leq \delta_\epsilon(d_k) \lambda(U)^{-1} \sum_{V \in \Omega_k} \lambda(V).$$

Since all $V \in \Omega_k$ must touch the compact support of ρ_ϵ and their diameters tend uniformly to 0 with k , their joint area remains bounded depending solely on U, ϵ . Since also d_k tend to 0 with k , for all $k \geq k(\epsilon)$,

$$\int_U (\tilde{\rho}_\epsilon(z) - \rho_{\epsilon,k}(z)) d\lambda(z) \leq \frac{2}{5} \eta.$$

Taking into account estimate (4), for $k \geq k(\epsilon)$, $\int_U \rho_{\epsilon,k}(z) d\lambda(z) \geq \eta/2$. Based on estimate (3), we conclude that for all $k \geq k(\epsilon)$,

$$(5) \quad \rho_{\epsilon,k}(z) \geq Q_1 > 0$$

for all $z \in U$ and Q_1 which only depends on U . Next, we estimate

$$\int_U (\rho_{\epsilon,k}(z) - \rho(z))_+ d\lambda(z) \leq \int_U (\tilde{\rho}_\epsilon(z) - \rho(z))_+ d\lambda(z) = \int_{\mathbb{C}} (\rho_\epsilon(w) - \rho(w))_+ d\lambda(w) < \epsilon$$

where we used a change of variables formula and condition (1). For every $\epsilon > 0$ and $k \geq k(\epsilon)$, we conclude from this and estimate (5) that $\rho(z) < \frac{Q_1}{2}$ on a set λ -measure less than $\frac{2\epsilon}{Q_1}$. Since ϵ can be made arbitrarily small while Q_1 is fixed, then $\rho(z) \geq \frac{Q_1}{2}$ on a set of full λ -measure in U . \square

2.1. Return times. Introduce the following function $g : \mathbb{R} \rightarrow \mathbb{R}$: $g(x) = |\Lambda| \sqrt{e^x}$.

Lemma 3. *There exists N_0 such that for all $n \geq N_0$, there exist sets $W_+, W_- \subset A_n$ which consist of fundamental rectangles each of which is mapped by f onto some $A_m \subset \{z \in \mathbb{C} : |z| \geq g(|\Lambda|e^n)\}$ in the case of W_+ , $A_m \subset \{z \in \mathbb{C} : |z| \leq g(-|\Lambda|e^n)\}$ for W_- and such that*

$$\lambda(W_\pm) > \frac{1}{4} \lambda(A_n).$$

Proof. For an annulus centered at 0 with inner radius r , $1/3$ of its area belongs to the half-plane $\Re z > r/2$ and another $1/3$ to $\Re z < -r/2$. For A_n with n large enough, almost the entire area, certainly more than $1/4$ of the area of the whole annulus, of $A_n \cap \{z : \Re z > |\Lambda| \exp n\}$ can be filled with fundamental rectangles. This defines W_+ . The set W_- is constructed in the same way. \square

The following lemma generalizes Lemma 3.

Lemma 4. *There are constants N_1 and $K_0 > 1$ such that for all $n \geq N_1$ and any integer $p \geq 1$, there is a set $W_p \subset A_n$ such that:*

- W_p is the union of sets each of which is mapped by f^{p-1} univalently onto a fundamental rectangle,
- for every $z \in W_p$ and $0 < j < p$, $f^j(z) \in A_m$ with $m \geq n$, while $f^p(z) \in A_m$ with $m \geq g^p(|\Lambda|e^n)$,
- $\lambda(W_p) \geq K_0^{-p}$.

Proof. Choose N_1 at least as large as N_0 in Lemma 3 and so large that $g(|\Lambda|e^{N_1}) \geq |\Lambda|e^{N_1}$. Additionally, the orbit of 0 must fit inside $D(0, |\Lambda| \exp(N_1 - 1))$. For $p = 1$ the claim follows from Lemma 3. Assuming now the claim for some $p \geq 1$, we can first split the set W_p into W_p^m , $A_m \subset \{z \in \mathbb{C} : |z| \geq g^p(|\Lambda|e^n)\}$, defined by $W_p^m = \{z \in W_p : f^p(z) \in A_m\}$. The set W_p^m splits into the union of topological disks each of which is initially mapped by f^{p-1} univalently onto a fundamental rectangle and then by f . Since by our choice of N_1 the post-singular set is far away surrounded by A_{N_1-1} , the first map has distortion bounded independently of m, p and the distortion of f satisfies the explicit bound of e . Let $Q > 1$ bound the ratio of the squares of the derivatives for any branch of f^{-p} from A_m into W_p^m at any two points of A_m . Since $m \geq N_1 \geq N_0$ by our choice of N_1 ,

inside A_m we can find W given by Lemma 3 and then define $W_{p+1}^m := W_p^m \cap f^{-p}(W)$. As a consequence of the bounded distortion of f^p and Lemma 3, $\frac{\lambda(W_{p+1}^m)}{\lambda(W_p^m)} \geq (4Q)^{-1}$. Now set $W_{p+1} = \bigcup_{m \geq g^p(n)} W_{p+1}^m$. Then $\lambda(W_{p+1}) \geq (4Q)^{-1} \lambda(W_p)$ so with $K_0 = 4Q$ the last claim of Lemma 3 will persist under induction. The remaining claims follow immediately from Lemma 3 and the construction of W_{p+1} . \square

Proposition 1. *There exist constants N_2 and $K_0, K_1 > 1$ such that for each $n \geq N_2$ and $p \geq 1$, A_n contains a subset V_p , such that V_p are pairwise disjoint for different p and for every $z \in V_p$, $|f^i(z)| \geq |\Lambda|e^n$ for $i = 0, \dots, p$ while $|f^{p+1}(z)| \leq g(-g^p(|\Lambda|e^n))$. Additionally, for each p , $\lambda(V_p) \geq K_1^{-1} K_0^{-p} \lambda(A_n)$.*

Proof of the Proposition. We choose N_2 at least equal to N_1 from Lemma 4, such that $g(|\Lambda|e^n) \geq |\Lambda|e^n$ if $n \geq N_2$ and so big that the orbit 0 fits inside $D(0, |\Lambda|e^{N_2-1})$ and at least 1. By the last choice, the pairwise disjointness of sets V_p will follow automatically from the conditions on orbits from V_p . Consider first the set W_p obtained from Lemma 4. It consists of sets U_j which are univalent preimages of fundamental rectangles, each of which is mapped with bounded distortion onto $A_m \subset \{z \in \mathbb{C} : |z| \geq g^p(|\Lambda|e^n)\}$. Thus, a portion of U_j of area at least $K_1^{-1} \lambda(U_j)$ with K_1 a constant, is occupied by the preimage by f^p of the set W_- from Lemma 3. It is immediate that every z from this preimage satisfies the demands of Proposition 1. V_p is the union of such preimages for all U_j and hence its measure is bounded below as claimed in the Proposition.

Proof of Theorem 1.

Lemma 5. *For all $x \geq N_3$ for some N_3 and every $\gamma > 0$, $\lim_{p \rightarrow \infty} g^p(x) \gamma^{-p} = +\infty$.*

Proof. Evidently, $g(x)/x$ tends to ∞ , so pick N_3 so that for all $x \geq N_3$, $g(x) \geq 2x$. Then $g^p(x) \geq 2^p x$ for all $p \geq 1$, in particular $g^p(x) - g^{p-1}(x) \rightarrow +\infty$. But $\frac{g^{p+1}(x)}{g^p(x)} = \exp(g^p(x) - g^{p-1}(x))$. \square

Consider a slit annulus A_n for n at least equal to the constant N_2 of Proposition 1 and $|\Lambda|e^n \geq N_3$ of Lemma 5. Let $\tau(z)$ for $z \in A_n$ be the first return time to A_n . Note that μ -almost every point returns since open sets return and μ is ergodic. Clearly τ is μ -integrable, but then also λ -integrable in view of Lemma 1. Similarly, λ -almost every point returns. If $z \in D(0, r)$ then it takes at least $k \geq K_2 \log r^{-1}$ for $f^k(z)$ to get in the distance at least 1 away from the orbit of 0. K_2 is a positive constant which depends on the maximum modulus of the derivative of f on some compact set. It follows that on each set V_p from Proposition 1, the return time is at least $K_2(\log |\Lambda| + g^p(|\Lambda|e^n))$. Since the measure of V_p is only exponentially small with p , by Lemma 5, the return time is not λ -integrable which gives us the final contradiction.

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